[2] [a] $-1 \leq \sin n \leq 1$ $1\frac{1}{2}$

$\Rightarrow -5 \leq 5\sin n \leq 5$ $1\frac{1}{2}$

$\Rightarrow -7 \leq 5\sin n - 2 \leq 3$ $1\frac{1}{2}$

$$\Rightarrow \frac{-7}{n(\ln n)^2} \leq \frac{5\sin n - 2}{n(\ln n)^2} \leq \frac{3}{n(\ln n)^2}$$ $1\frac{1}{2}$

$$\lim_{n\to\infty} \frac{-7}{n(\ln n)^2} = \lim_{n\to\infty} \frac{3}{n(\ln n)^2} = 0$$

So, by Squeeze Theorem, $\lim\limits_{n\to\infty} \dfrac{5\sin n - 2}{n(\ln n)^2} = 0$

So, the Divergence Test cannot be used to determine whether the series converges or diverges

ALL MARKED ITEMS $2\frac{1}{2}$ POINTS EACH UNLESS OTHERWISE INDICATED

[b] $-7 \leq 5\sin n - 2 \leq 3$

$\Rightarrow 0 \leq |5\sin n - 2| \leq 7$ 3½

$\Rightarrow 0 \leq \frac{|5\sin n - 2|}{n(\ln n)^2} \leq \frac{7}{n(\ln n)^2}$ 1½ $n > 2$, and since $n \neq 1$, therefore $\ln n \neq 0$

Let $f(x) = \frac{7}{x(\ln x)^2}$ for $x \geq 2$

1½ $f(x) > 0$ since $x \geq 2 > 1$, therefore $\ln x > 0$

$f(x)$ is continuous on $(1, \infty)$, so $f(x)$ is continuous on $[2, \infty)$

3½

$f'(x) = -\frac{7}{[x(\ln x)^2]^2}((\ln x)^2 + x(2\ln x)\frac{1}{x}) = -\frac{7((\ln x)^2 + 2\ln x)}{x^2(\ln x)^4} > 0$ 1½

7

So, $f(x)$ is decreasing on $[2, \infty)$ 1½

$$\int_2^\infty \frac{7}{x(\ln x)^2}dx = 7\lim_{N\to\infty}\int_2^N \frac{1}{x(\ln x)^2}dx$$

$u = \ln x \Rightarrow \int \frac{1}{u^2}du = -\frac{1}{u} = -\frac{1}{\ln x}$

$$= 7\lim_{N\to\infty}\left(-\frac{1}{\ln x}\right)\Bigg|_2^N$$ 1½

$$= -7\lim_{N\to\infty}\left(\frac{1}{\ln N} - \frac{1}{\ln 2}\right)$$

$$= \frac{7}{\ln 2}$$

Since $\int_2^\infty \frac{7}{x(\ln x)^2}dx$ converges, therefore $\sum_{n=2}^\infty \frac{7}{n(\ln n)^2}$ converges (Integral Test) 1½

therefore $\sum_{n=2}^\infty \frac{|5\sin n - 2|}{n(\ln n)^2}$ converges (Comparison Test) 1½

therefore $\sum_{n=2}^\infty \frac{5\sin n - 2}{n(\ln n)^2}$ converges absolutely (Absolute Convergence Test) 1½

[3]

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3n+3)(4+3x)^{n+1}}{2(n+1)^2+1}\frac{2n^2+1}{3n(4+3x)^n}\right| \quad 3\tfrac{1}{2}$$

$$= \lim_{n\to\infty}\left|\frac{3+\frac{3}{n}}{3}\,\frac{2+\frac{1}{n^2}}{2(\frac{1}{1+\frac{1}{n}})^2+\frac{1}{n^2}}(4+3x)\right| \quad 8$$

$$= 1\cdot 1\cdot|(4+3x)| \quad 3\tfrac{1}{2}$$

$$= |4+3x|$$

$1\tfrac{1}{2}$ $|4+3x|<1 \;\Rightarrow\; -1 < 4+3x < 1$ $1\tfrac{1}{2}$

$$\Rightarrow -5 < 3x < -3$$

$$\Rightarrow -\tfrac{5}{3} < x < -1 \quad 1\tfrac{1}{2}$$

$x=-1$:

$$\sum_{n=1}^{\infty}\frac{3n(4+3(-1))^n}{2n^2+1} = \sum_{n=1}^{\infty}\frac{3n(1)^n}{2n^2+1} = \sum_{n=1}^{\infty}\frac{3n}{2n^2+1}$$

$$3\tfrac{1}{2}\quad \frac{3n}{2n^2+1} > \frac{3n}{2n^2+n^2} = \frac{3n}{3n^2} = \frac{1}{n} > 0 \quad 1\tfrac{1}{2}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (Harmonic Series), therefore $\sum_{n=1}^{\infty}\frac{3n-1}{2n^2+1}$ diverges (Comparison Test)

$1\tfrac{1}{2}$ $1\tfrac{1}{2}$

$x=-\frac{5}{3}$:

$$\sum_{n=1}^{\infty}\frac{3n(4+3(-\frac{5}{3}))^n}{2n^2+1} = \sum_{n=1}^{\infty}\frac{3n(-1)^n}{2n^2+1} = \sum_{n=1}^{\infty}(-1)^n\frac{3n}{2n^2+1}$$

$$a_n = \frac{3n}{2n^2+1} > 0$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\frac{3}{n}}{2+\frac{1}{n^2}} = 0 \quad 1\tfrac{1}{2}$$

Let $f(x) = \dfrac{3x}{2x^2+1}$

$$f(x) = \frac{3(2x^2+1)-(3x)(4x)}{(2x^2+1)^2} = \frac{3-6x^2}{(2x^2+1)^2} = -\frac{3(2x^2-1)}{(2x^2+1)^2} < 0 \quad 7$$

since $x\ge 1$, $1\tfrac{1}{2}$

therefore $x^2\ge 1$,
so $2x^2\ge 2$,
so $2x^2-1\ge 1>0$

So, $\{a_n\}$ is decreasing $1\tfrac{1}{2}$

So, $\sum_{n=1}^{\infty}(-1)^n\frac{3n}{2n^2+1}$ converges (Alternating Series Test)

$1\tfrac{1}{2}$

So, the interval of convergence of $\sum_{n=1}^{\infty}\frac{(3n-1)(4+3x)^n}{2n^2+1}$ is $[-\frac{5}{3},-1)$.

[4]

$$\lim_{n\to\infty}\sqrt[n]{\left|(1-2\tan\tfrac{1}{n})^{n^2}\right|} = \lim_{n\to\infty}\left|(1-2\tan\tfrac{1}{n})^{n}\right|$$

$$= \lim_{n\to\infty}(1-2\tan\tfrac{1}{n})^{n} \quad 1\tfrac{1}{2} \qquad \text{since } \lim_{n\to\infty}(1-2\tan\tfrac{1}{n}) = 1-2\tan 0 = 1 > 0$$

$$= e^{\lim_{n\to\infty} n\ln(1-2\tan\frac{1}{n})}$$

$$= e^{-2} < 1$$

So $\sum_{n=1}^{\infty}(1-2\tan\tfrac{1}{n})^{n^2}$ converges (Root Test)

$1\tfrac{1}{2}$

$$\lim_{n\to\infty} n\ln(1-2\tan\tfrac{1}{n}) = \lim_{n\to\infty}\frac{\ln(1-2\tan\frac{1}{n})}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{1-2\tan\frac{1}{n}}(-2\sec^2\frac{1}{n})(-\frac{1}{n^2})}{-\frac{1}{n^2}} \quad 7$$

$$= \lim_{n\to\infty}\frac{-2\sec^2\frac{1}{n}}{1-2\tan\frac{1}{n}}$$

$$= \frac{-2(1)}{1-2(0)}$$

$$= -2$$